88087201

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Friday 7 November 2008 (afternoon)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 6]*

When $f(x) = x^4 + 3x^3 + px^2 - 2x + q$ is divided by $(x-2)$ the remainder is 15, and $(x+3)$ is a factor of $f(x)$.

Find the values of p and q.

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2. *[Maximum mark: 5]*

Write $\ln(x^2-1)-2\ln(x+1)+\ln(x^2+x)$ as a single logarithm, in its simplest form.

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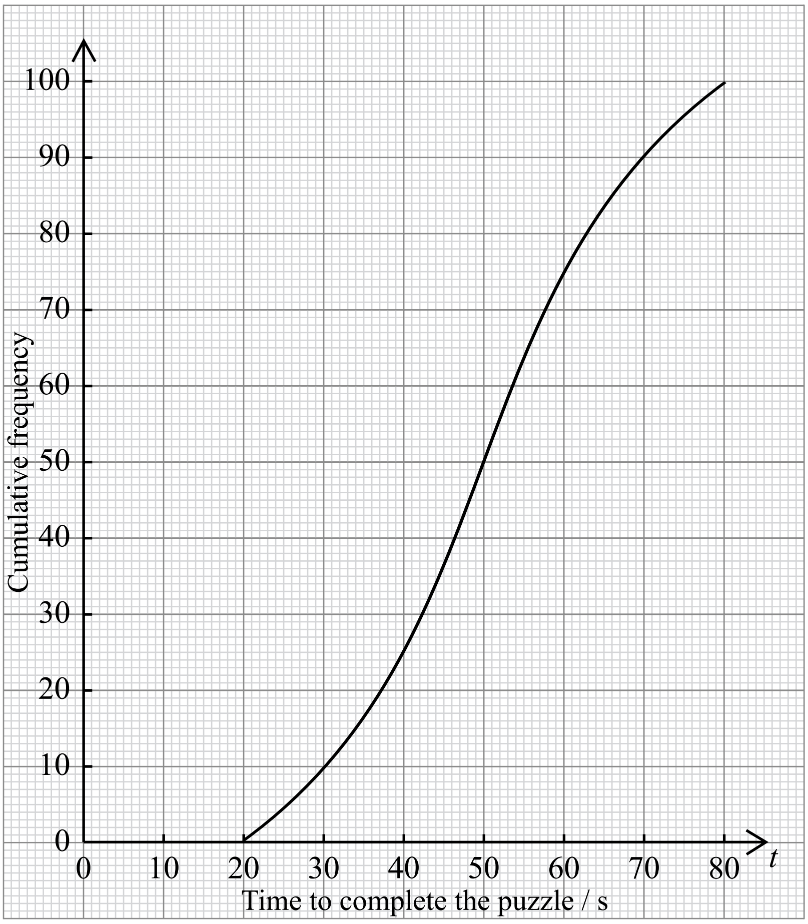
3. *[Maximum mark: 5]*

A recruitment company tests the aptitude of 100 applicants applying for jobs in engineering. Each applicant does a puzzle and the time taken, t, is recorded. The cumulative frequency curve for these data is shown below.



Using the cumulative frequency curve,

(a) write down the value of the median; *[1 mark]*

(b) determine the interquartile range; *[2 marks]*

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(This question continues on the following page)

(Question 3 continued)

(c) complete the frequency table below. *[2 marks]*

Time to complete puzzle in seconds	**Number of applicants**
$20 < t \le 30$	
$30 < t \le 35$	
$35 < t \le 40$	
$40 < t \le 45$	
$45 < t \le 50$	
$50 < t \le 60$	
$60 < t \le 80$	

4. *[Maximum mark: 4]*

An 81 metre rope is cut into n pieces of increasing lengths that form an arithmetic sequence with a common difference of d metres. Given that the lengths of the shortest and longest pieces are 1.5 metres and 7.5 metres respectively, find the values of n and d.

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5. *[Maximum mark: 5]*

Calculate the exact value of $\int_1^{e} x^2 \ln x \, dx$.

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6. *[Maximum mark: 7]*

Find the equation of the normal to the curve $5xy^2 - 2x^2 = 18$ at the point $(1, 2)$.

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7. *[Maximum mark: 7]*

(a) Use the derivatives of $\sin x$ and $\cos x$ to show that the derivative of $\tan x$ is $\sec^2 x$. *[3 marks]*

(b) Hence by using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$, show that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$. *[4 marks]*

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8. *[Maximum mark: 7]*

John removes the labels from three cans of tomato soup and two cans of chicken soup in order to enter a competition, and puts the cans away. He then discovers that the cans are identical, so that he cannot distinguish between cans of tomato soup and chicken soup. Some weeks later he decides to have a can of chicken soup for lunch. He opens the cans at random until he opens a can of chicken soup. Let Y denote the number of cans he opens.

Find

(a) the possible values of Y, *[1 mark]*

(b) the probability of each of these values of Y, *[4 marks]*

(c) the expected value of Y. *[2 marks]*

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9. *[Maximum mark: 8]*

A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and bottom of the box are identical regular hexagons of side x cm.

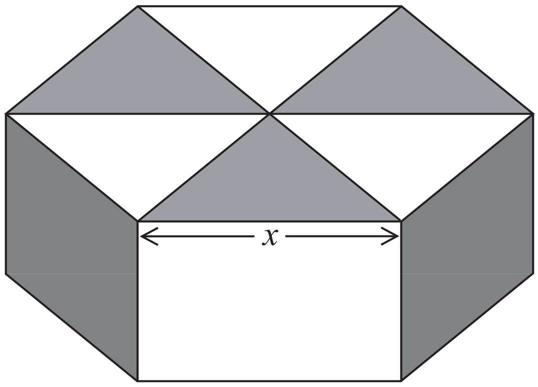

diagram not to scale

(a) Show that the area of each hexagon is $\frac{3\sqrt{3}x^2}{2}$ cm^2. *[1 mark]*

(b) Given that the volume of the box is 90 cm^3, show that when $x = \sqrt[3]{20}$ the total surface area of the box is a minimum, justifying that this value gives a minimum. *[7 marks]*

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10. *[Maximum mark: 6]*

Three distinct non-zero vectors are given by $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, and $\overrightarrow{OC} = \boldsymbol{c}$.

If $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$ and $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CA}$, show that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$.

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 21]*

(a) Sketch the curve $f(x) = \sin 2x$, $0 \le x \le \pi$. *[2 marks]*

(b) Hence sketch on a separate diagram the graph of $g(x) = \csc 2x$, $0 \le x \le \pi$, clearly stating the coordinates of any local maximum or minimum points and the equations of any asymptotes. *[5 marks]*

(c) Show that $\tan x + \cot x \equiv 2\csc 2x$. *[3 marks]*

(d) Hence or otherwise, find the coordinates of the local maximum and local minimum points on the graph of $y = \tan 2x + \cot 2x$, $0 \le x \le \frac{\pi}{2}$. *[5 marks]*

(e) Find the solution of the equation $\csc 2x = 1.5\tan x - 0.5$, $0 \le x \le \frac{\pi}{2}$. *[6 marks]*

12. *[Maximum mark: 14]*

(a) Using mathematical induction, prove that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}, \ n \in \mathbb{Z}^+.$$

[9 marks]

(b) Show that the result holds true for $n = -1$. *[5 marks]*

13. *[Total mark: 25]*

Part A *[Maximum mark: 12]*

(a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$. *[6 marks]*

(b) Draw these roots on an Argand diagram. *[2 marks]*

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + \mathrm{i}b$. *[4 marks]*

Part B *[Maximum mark: 13]*

(a) Expand and simplify $(x-1)(x^4 + x^3 + x^2 + x + 1)$. *[2 marks]*

(b) Given that b is a root of the equation $z^5 - 1 = 0$ which does not lie on the real axis in the Argand diagram, show that $1 + b + b^2 + b^3 + b^4 = 0$. *[3 marks]*

(c) If $u = b + b^4$ and $v = b^2 + b^3$ show that

(i) $u + v = uv = -1$;

(ii) $u - v = \sqrt{5}$, given that $u - v > 0$. *[8 marks]*